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ON LINEAR RELATIONS BETWEEN L-VALUES AND ARITHMETIC FUNCTIONS (Analytic and Arithmetic Theory of Automorphic Forms)

AUTHOR(S):

Su, Ren-He

CITATION:

Su, Ren-He. ON LINEAR RELATIONS BETWEEN L-VALUES AND ARITHMETIC FUNCTIONS (Analytic and Arithmetic Theory of Automorphic Forms). 数理解析研究所講究録 2019, 2100: 14-20

ISSUE DATE:

2019-01

URL:

<http://hdl.handle.net/2433/251789>

RIGHT:

ON LINEAR RELATIONS BETWEEN L-VALUES AND ARITHMETIC FUNCTIONS

REN-HE SU

1. RESULTS FROM SIEGEL

Let $F = \mathbb{Q}(\sqrt{D})$ be a quadratic field with discriminant $D > 0$, ring of integers \mathfrak{o} and different \mathfrak{d} . For integer $k \geq 2$ and B an ideal class of F , we define the Eisenstein series

$$E_{k,B}(z) = \frac{1}{4}\zeta_{\mathfrak{d}B}(1-k) + \sum_{\substack{\xi \in \mathfrak{d}^{-1} \\ \xi > 0}} \sigma_{k-1,\mathfrak{d}B}(\xi\mathfrak{d})q^{\xi}$$

of weight k and level $\mathrm{SL}_2(\mathfrak{o})$, where $z \in \mathfrak{h}^2$ and $q^{\xi} = \exp(2\pi\sqrt{-1}\mathrm{Tr}(\xi z))$. If we let

$$\mathcal{R}E_{k,B}(\tau) = E_{k,B}(\tau, \tau)$$

for $\tau \in \mathfrak{h}$, then $\mathcal{R}E_{k,B} \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. It is easy to see that

$$\mathcal{R}E_{k,B}(\tau) = \frac{1}{4}\zeta_{\mathfrak{d}B}(1-k) + \sum_{n=1}^{\infty} \left(\sum_{\substack{\xi \in \mathfrak{d}^{-1} \\ \xi > 0 \\ \mathrm{Tr}(\xi) = n}} \sigma_{k-1,\mathfrak{d}B}(\xi\mathfrak{d}) \right) q^n,$$

where $q^n = \exp(2\pi\sqrt{-1}n\tau)$. Thus by comparing the Fourier coefficients, we have the following results.

Theorem 1 (Siegel). *If $k = 2, 4$, then*

$$\zeta_B(1-k) = -\frac{B_{2k}}{k} \sum_{\substack{\xi \in \mathfrak{d}^{-1} \\ \xi > 0 \\ \mathrm{Tr}(\xi) = 1}} \sigma_{k-1,B}(\xi\mathfrak{d}).$$

Corollary 1. *We have*

$$\zeta_F(-1) = \frac{1}{60} \sum_{\substack{m \in \mathbb{Z} \\ m^2 < D \\ m^2 \equiv D \pmod{4}}} \sigma_1\left(\frac{D-m^2}{4}\right)$$

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and

$$\zeta_F(-3) = \frac{1}{120} \sum_{\substack{m \in \mathbb{Z} \\ m^2 < D \\ m^2 \equiv D \pmod{4}}} \sigma_3\left(\frac{D - m^2}{4}\right).$$

Today, we want to use the same technique on the Hilbert forms of half-integral weight.

2. MAIN RESULTS

Consider the same F as in the last section. Put

$$\omega = \begin{cases} \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod{4}, \\ \frac{\sqrt{D}}{2} & \text{otherwise.} \end{cases}$$

Then $\mathfrak{o} = \mathbb{Z} + \omega\mathbb{Z}$. For any two ideals \mathfrak{b} and \mathfrak{c} of F such that $\mathfrak{b}\mathfrak{c} \subset \mathfrak{o}$, we let

$$\Gamma[\mathfrak{b}, \mathfrak{c}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \mid a, d \in \mathfrak{o}, b \in \mathfrak{b}, c \in \mathfrak{c} \right\}$$

and put $\Gamma = \Gamma[\mathfrak{d}^{-1}, 4\mathfrak{d}]$.

Definition 1. For any $z \in \mathfrak{h}^2$, we put

$$\theta(z) = \sum_{\xi \in \mathfrak{o}} q^{\xi^2} \quad (z \in \mathfrak{h}^2)$$

The factor of automorphy of weight $1/2$ is given by

$$\tilde{j}(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}$$

where $\gamma \in \Gamma$.

Let $k \geq 0$. A Hilbert modular form of parallel weight $k + 1/2$ and level Γ is a holomorphic function f on \mathfrak{h}^2 which satisfies

$$f(\gamma z) = \tilde{j}(\gamma, z)^{2k+1} f(z)$$

for any $\gamma \in \Gamma$. The space of all such forms is denoted by $M_{k+1/2}(\Gamma)$ and the subspace consisting cusp forms in it is denoted by $S_{k+1/2}(\Gamma)$.

Now suggest that the different \mathfrak{d} has a totally positive generator δ . Put $\Gamma' = \Gamma[\mathfrak{o}, 4\mathfrak{o}]$. A Hilbert modular form of parallel weight $k + 1/2$ and level Γ' is a function with the form

$$f_0(z) = f(\delta^{-1}z)$$

where $f \in M_{k+1/2}(\Gamma)$. It satisfies the automorphic condition

$$f_0(\gamma z) = \tilde{j}_0(\gamma, z)^{2k+1} f_0(z)$$

where $\gamma \in \Gamma'$ and \tilde{j}_0 comes from θ_0 as for \tilde{j} . The space of all such forms and cusp forms are denoted by $M_{k+1/2}(\Gamma')$ and $S_{k+1/2}(\Gamma')$, respectively.

Note that the congruence subgroup $\Gamma_0(4) \subset \mathrm{SL}_2(\mathbb{Z})$ can be embedded into Γ' diagonally. One can show that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we have

$$\tilde{j}_0(\gamma, (\tau, \tau)) = (c\tau + d)\chi_{-4}(d)$$

where $\tau \in \mathfrak{h}$. Thus if we put

$$\mathcal{R}f(\tau) = f_0((\tau, \tau))$$

where $f \in M_{k+1/2}(\Gamma)$ and $\tau \in \mathfrak{h}$, then

$$\mathcal{R}f \in M_{2k+1}(\Gamma_0(4), \chi_{-4}).$$

We write δ in the form

$$\delta = (\alpha + \beta\omega)\sqrt{D}$$

with $\alpha + \beta\omega > 0$ a unit of norm -1 . Then if f has the Fourier expansion

$$f(z) = \sum_{\xi \in \mathfrak{o}} c(\xi)q^\xi \quad (z \in \mathfrak{h}^2),$$

we can write down the Fourier expansion of $\mathcal{R}f$ explicitly as

$$\mathcal{R}f(\tau) = \sum_{n=0}^{\infty} \left(\sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ a\beta - b\alpha = n}} c(a + b\omega) \right) q^n \quad (\tau \in \mathfrak{h}).$$

If we apply the mapping \mathcal{R} on θ^2 , we get the following result.

Corollary 2. *For $m > 0$ and number field K we set*

$$r_{K,m}(x) = \# \left\{ (x_1, x_2, \dots, x_m) \in \mathfrak{o}_K^m \mid x_1^2 + \dots + x_m^2 = x \right\} \quad (x \in K).$$

Then with the notations given above, we have

$$\sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ a\beta - b\alpha = n}} r_{F,m}(a + b\omega) = r_{\mathbb{Q},2m}(n).$$

From now let us consider the case for Kohnen plus space. The specific spaces were first defined by Kohnen [3] in 1980 and generalized to the case for Hilbert modular forms by Hiraga and Ikeda [2] in 2013. For any $\xi \in F$, we denote by

$$\xi \equiv \square \pmod{4}$$

if there exists $\lambda \in \mathfrak{o}$ such that $\xi - \lambda^2 \in 4\mathfrak{o}$.

Definition 2. *The Kohnen plus space is a subspace of $M_{k+1/2}(\Gamma)$ defined as*

$$M_{k+1/2}^+(\Gamma) = \left\{ f(z) = \sum_{\xi} c(\xi) q^{\xi} \in M_{k+1/2}(\Gamma) \mid c(\xi) = 0 \text{ unless } (-1)^k \xi \equiv \square \pmod{4} \right\}.$$

Also we put $S_{k+1/2}^+(\Gamma) = M_{k+1/2}^+(\Gamma) \cap S_{k+1/2}(\Gamma)$. Their images in $M_{k+1/2}(\Gamma')$ under the isomorphism $f \mapsto f_0$ are denoted by $M_{k+1/2}^+(\Gamma')$ and $S_{k+1/2}^+(\Gamma')$, respectively.

We also define the plus spaces contained in $M_{2k+1}(\Gamma_0(4), \chi_{-4})$ and $S_{2k+1}(\Gamma_0(4), \chi_{-4})$.

Definition 3. *We put*

$$M_{2k+1}^+(\Gamma_0(4), \chi_{-4}) = \left\{ h(z) = \sum_{n=0}^{\infty} d(n) q^n \in M_{2k+1}(\Gamma_0(4), \chi_{-4}) \mid d(n) = 0 \text{ if } \chi_{-4}(n) = (-1)^{k+1} \right\}$$

and $S_{2k+1}^+(\Gamma_0(4), \chi_{-4}) = M_{2k+1}^+(\Gamma_0(4), \chi_{-4}) \cap S_{2k+1}(\Gamma_0(4), \chi_{-4})$.

For $k > 0$ being odd, the space $S_{2k+1}^+(\Gamma_0(4), \chi_{-4})$ was defined by Kojima [4] in 1982 and shown to be isomorphic to $M_{2k+2}(\Gamma^2(\mathcal{O}))$, the space of Hermitian modular forms of weight $2k+2$ and degree 2. The plus spaces of odd weights can be described exactly in the sense taking certain linear combinations of the normalized Hecke eigenforms of the whole space as a basis.

Theorem 2. *Let $k \geq 0$. We have*

$$\mathcal{R} \left(M_{k+1/2}^+(\Gamma) \right) \subset M_{2k+1}^+(\Gamma_0(4), \chi_{-4})$$

and

$$\mathcal{R} \left(S_{k+1/2}^+(\Gamma) \right) \subset S_{2k+1}^+(\Gamma_0(4), \chi_{-4})$$

This theorem can be proved in an elementary way, but can also be proved in a representation theoretical way, which reflects the nature of both plus spaces of half-integral and integral weights more.

As an application, we apply \mathcal{R} on the Eisenstein series in $M_{k+1/2}^+(\Gamma)$, which was introduced in [5]. Let χ be a character of the ideal class group $Cl(F)$ of F . The Eisenstein series in $M_{k+1/2}^+(\Gamma)$ with respect to

χ is given by

$$E_{k+1/2,\chi}(z) = L_F(1-2k, \bar{\chi}^2) + \sum_{\substack{(-1)^k \xi \equiv \square \pmod{4} \\ \xi > 0}} \mathcal{H}_k(\xi, \chi) q^\xi$$

where

$$\begin{aligned} \mathcal{H}_k(\xi, \chi) &= \chi(\mathcal{D}_{(-1)^k \xi}) L_F(1-k, \chi_{(-1)^k \xi} \chi) \\ &\times \sum_{\mathfrak{a} | \mathfrak{f}_{(-1)^k \xi}} \mu_F(\mathfrak{a}) \chi_{(-1)^k \xi}(\mathfrak{a}) \chi(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a})^{k-1} \sigma_{F, 2k-1, \chi^2}(\mathfrak{f}_{(-1)^k \xi} \mathfrak{a}^{-1}). \end{aligned}$$

Here \mathcal{D}_x and χ_x are the relative discriminant and the quadratic character associated to $F(\sqrt{x})/F$, respectively, and \mathfrak{f}_x is the integral ideal such that $\mathfrak{f}_x^2 \mathcal{D}_x = (x)$. Further more, μ_F is the Möbius function with respect to F and

$$\sigma_{F, m, \chi'}(\mathfrak{b}) = \sum_{\mathfrak{r} | \mathfrak{b}} N_{F/\mathbb{Q}}(\mathfrak{r})^m \chi'(\mathfrak{r}).$$

The Eisenstein series given above is a generalization of the one given by Cohen in [1], whose Fourier coefficients are called generalized Hurwitz class numbers.

The two Eisenstein series in $M_{2k+1}(\Gamma_0(4), \chi_{-4})$ are given by

$$E_{2k+1, \chi_{-4}}(\tau) = 1 + \frac{2}{L(-2k, \chi_{-4})} \sum_{n=1}^{\infty} \sigma_{2k, \chi_{-4}}(n) q^n$$

and

$$F_{2k+1, \chi_{-4}}(\tau) = \frac{(-1)^k 2}{L(-2k, \chi_{-4})} \sum_{n=1}^{\infty} \sigma'_{2k, \chi_{-4}}(n) q^n$$

where

$$\sigma'_{2k, \chi_{-4}}(n) = \sum_{r | n} r^{2k} \chi_{-4}(n/r).$$

The series $F_{2k+1, \chi_{-4}}$ is the normalized image of $E_{2k+1, \chi_{-4}}$ under the Fricke involution. By comparison of the constant term at the cusps of $\Gamma_0(4)$, we have the following result.

Theorem 3. *For $k \geq 0$, we have*

$$\mathcal{R}G_{k+1/2, \chi} - L_F(1-2k, \bar{\chi}^2)(E_{2k+1, \chi_{-4}} + (-1)^k F_{2k+1, \chi_{-4}}) \in S_{2k+1}(\Gamma_0(4), \chi_{-4}).$$

In particular, since $S_3(\Gamma_0(4), \chi_{-4}) = 0$, we have

$$\sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ a\beta - b\alpha = n}} \mathcal{H}_1(a + b\omega, \chi) = -4L_F(1-2k, \bar{\chi}^2)(\sigma_{2, \chi_{-4}}(n) - \sigma'_{2, \chi_{-4}}(n)).$$

For example, let $F = \mathbb{Q}(\sqrt{5})$ and $\delta = \omega\sqrt{5}$ where $\omega = (1 + \sqrt{5})/2$. Then by applying the theorem on $E_{3/2,1}$ and $E_{5/2,1}$ we may get

$$\begin{aligned}
L_F(0, \chi_{-2-\omega}) &= -\frac{2}{15} (\sigma_{2, \chi_{-4}}(2) - \sigma'_{2, \chi_{-4}}(2)) , \\
2L_F(0, \chi_{-3}) + 2L_F(0, \chi_{-3+\omega}) &= -\frac{2}{15} (\sigma_{2, \chi_{-4}}(3) - \sigma'_{2, \chi_{-4}}(3)) , \\
2L_F(0, \chi_{-4}) &= -\frac{2}{15} (\sigma_{2, \chi_{-4}}(4) - \sigma'_{2, \chi_{-4}}(4)) , \\
2L_F(0, \chi_{-3}) + 2L_F(0, \chi_{-6-\omega}) &= -\frac{2}{15} (\sigma_{2, \chi_{-4}}(6) - \sigma'_{2, \chi_{-4}}(6)) , \\
2\zeta_F(-1) &= \frac{1}{75} (\sigma_{4, \chi_{-4}}(1) + \sigma'_{4, \chi_{-4}}(1) + 3s(1)) , \\
4\zeta_F(-1) + 2L_F(-1, \chi_{5+\omega}) &= \frac{1}{75} (\sigma_{4, \chi_{-4}}(5) + \sigma'_{4, \chi_{-4}}(5) + 3s(5)) , \dots
\end{aligned}$$

and so on. Here

$$s(n) = \frac{1}{4} \sum_{a^2+b^2=n} (a + b\sqrt{-1})^4.$$

3. REPRESENTATION THEORETIC VIEW OF THE THEOREM

Let $\psi_0 = \prod_{0, v \leq \infty} \psi_{0,v}$ be the additive character of \mathbb{A}_F/F such that for $v \mid \infty$ it satisfies

$$\psi_{0,v}(x) = \exp(2\pi\sqrt{-1}(-1)^k x) \quad (x \in \mathbb{R}).$$

Put $\psi = \psi_0(\delta^{-1} \cdot)$. We denote the metaplectic double covering of SL_2 by Mp_2 , which is with respect to the Kubota 2-cocycle. There exists an irreducible representation Ω_ψ of $\prod_{v \mid 2} \mathrm{Mp}_2(\mathfrak{o}_v)$ associated to ψ . This representation is a subquotient of the restricted Weil representation associated to ψ and is of 4-dimension. Note that a modular form of weight $k+1/2$ can be lifted to an automorphic form on $\mathrm{SL}_2(F) \backslash \mathrm{Mp}_2(\mathbb{A}_F)$. Hiraga and Ikeda [?] showed the following theorem

Theorem 4 (Hiraga, Ikeda). *Let $f_0 \in M_{k+1/2}(\Gamma')$ and $W_4 f_0$ be its image under the fricke involution with respect to $z \mapsto -\frac{1}{4z}$. A sufficient necessary condition for f_0 to be in the plus space is*

$$\langle \rho(\gamma) W_4 f_0 \mid \gamma \in \prod_{v \mid 2} \mathrm{Mp}_2(\mathfrak{o}_v) \rangle \cong \Omega_\psi.$$

Regardless of the choice of F , $\mathrm{SL}_2(\mathbb{Z}_2)$ can be embedded into $\prod_{v|2} \mathrm{Mp}_2(\mathfrak{o}_v)$ and the restriction of Ω_ψ to $\mathrm{SL}_2(\mathbb{Z}_2)$ remains the same. One can show

$$\Omega_\psi \Big|_{\mathrm{SL}_2(\mathbb{Z}_2)} = \pi_k \oplus \sigma_k$$

where π_k and σ_k are irreducible representations of dimension 3 and 1, respectively. They only depend on the parity of k . The following theorem was shown by Kojima [4] (for odd k , but the proof can be easily extended to general positive integer k).

Theorem 5 (Kojima). *Let $h \in M_{2k+1}(\Gamma_0(4), \chi_{-4})$. A sufficient necessary condition for h to be in the plus space is*

$$< \rho(\gamma)W_4h \mid \gamma \in \mathrm{SL}_2(\mathbb{Z}_2) > \cong \pi_k.$$

By Theorem 4, it is easy to see that if $f \in M_{k+1/2}^+(\Gamma)$ then $W_4\mathcal{R}f$ generates a irreducible representation of $\mathrm{SL}_2(\mathbb{Z}_2)$ equivalent to π_k . Thus we get $\mathcal{R}f \in M_{2k+1}^+(\Gamma_0(4), \chi_{-4})$.

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SCHOOL OF MATHEMATICAL SCIENCES, SICHUAN NORMAL UNIVERSITY, CHENGDU, SICHUAN, CHINA

Email address: tolo marc@ms57.hinet.net